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A Discrete – Time Mathematical Model for Homogeneous Population Density Dynamics of Single Weed Species

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Abstract

In this paper, the biological process is utilized to formulate a discrete-time homogeneous model for the dynamics of weed population density. Steady state solutions were obtained and analyzed them for local and global stabilities. The results revealed that our model is locally asymptotically stable but globally unstable. This result is contrary to the interesting property of the most standard biological one-dimensional discrete models, which display global stability if they are locally stable. Although, our model equation falls within the category of population models that exhibit local stability but globally not stable. It is concluded that, the weed population may exhibit unexpected behaviours.

Keywords: Biological process, Discrete-time model, Local stability, Global stability, Population density

1. Introduction

Weed is generally defined as uncultivated plant species that proliferate in agricultural setting thereby, interfering with crop production. In fact, weed is a term applied to any plant that grows naturally in a place it is not wanted. It exists only in natural environments that have been disturbed by humans such as agricultural lands, recreational parks, and irrigation dams (Akobundu 1987). Weeds form an important part of the land ecosystem, providing food and cover for animals and birds which are an important indicator of biodiversity health (Parsons, Benjamin, Clarke, Ginsburg, Mayes, Milne, and Wilkinson, 2009).

Population dynamics involve the study of population numerical change in time. The objectives are to identify the causes of numerical change in population and to explain how this cause act and interact to produce the observed pattern (Akinwande, Nasir and Abdulrahma, 2012). Over the years population models were concentrated mainly on the use of differential equations, even though, most populations such as weeds and phylogenetically more evolved organisms live in seasonal environments and because of this, have annual rhythms of reproduction and death. Continuous-time equations are not well suited to these kinds of processes. Discrete-time models are better suited for organism with annual or seasonal reproductive patterns (Allen, Allen, and Ponweeram, 1996; David, 1997; Alsharawi and Rhouma 2010; Sacker, 2010). Furthermore, many researchers have paid attention in recent times to discrete- time population models, since the discrete time models governed by discrete systems are more appropriate than the continuous ones when the populations have non over lapping generations (Wu and Zhang, 2014).

It is well-established that population models can be derived from two different sources, data and biological process. The first relies completely upon data to look at the dynamics of the population. (Burgess, 2011). The second is a model defined by biological processes, which do not include any data, but instead attempt to understand the dynamics of populations purely from what is expected to occur.

In this study biological process was employed to formulate discrete-time models for the dynamics of homogeneous weed population density

2.0 Material and Methods

2.1 Formulation of the Model Equations

The following assumptions are considered in the formulation of the model equations;

1. There are enough basic growth resources; e.g. nutrients, light, and water, that promote continuous growth of at least two plant species.
2. Within the populations of weed there are intra-specific competitions.
3. All parameters involved with the model formulation are non-negatives.

The figure below depicts the life cycle of weed

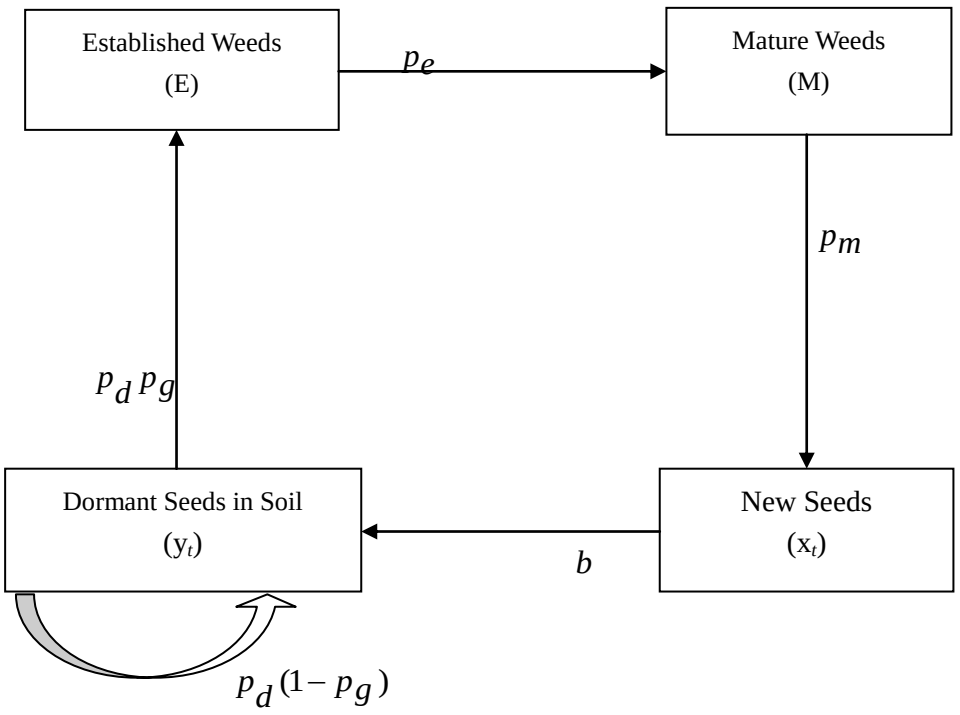

Figure 2.1 Schematic diagram for Population Cycle of Annual Weeds

Applying the assumptions, definition of variables and parameters as depicted in Figure 2.1 above, the difference equations which describe the dynamics of proliferation of single weed species is derived below.

The dormant seeds (y_t) in the year *t* consist of seeds from previous years (*t* - 1) that have survived the dry season and remained dormant (y_{t-1}), as well as new (fresh) seeds that have not germinated but viable (surviving) (x_{t-1}). Thus y_t satisfies the equation

$$y_t = p_d\left(1-p_g\right)\left(x_{t-1}+y_{t-1}\right) \tag{2.1}$$

Maximum weed seed pool (S_t) in year *t* is

$$S_t = x_t + y_t \tag{2.2}$$

The established weed densities *N* in the year t+1 is obtained from seed pool S_t in the top soil that survived the dry

season, germinate and become established. This is describe by

$$N_{t+1} = p_d p_g p_e S_t = p_d p_g p_e (x_t + y_t) \tag{2.3}$$

This is because the germination and establishment of new and residual seeds are two independent events.

The new seeds x_t produced in year *t* are a function of established matured weeds in the year *t* that is

$$x_t = f(p_m, N_t) = p_m \frac{bN_t}{1+aN_t} \tag{2.4}$$

Equation (2.4) is adaptation of Beaverton-Holts type function, because we assumed a density – dependent growth.

The function $f(x) = f(p_m, N_t)$ is expected to satisfy the following assumptions:

(i) f is a non-decreasing, bounded and continuous function on [0,∞)

(ii) $f(x) > 0$ for $x > 0$, $f(0) = 0$

(iii) *if* $f \in C^2([0,\infty))$ then $f'(x) > 0$ and $f''(x) < 0$ *for* $x > 0$.

This function satisfies conditions (i), (ii) and (iii)

These assumptions are established thus; we let $x = N_t$ in equation (2.4), so,

$f(x) = p_m \frac{bx}{1+ax}$, then, theorem 2.1 is employed to prove (i) and (ii).

Theorem 2.1 : $f(x)$ is non-decreasing, bounded and continuous function on [0,∞) if $x > y$ implies $f(x) > f(y)$, vice versa, $\forall\ x, y \in [0,\infty)$.

Proof : Suppose $f(x) > f(y)$, that is

$$p_m \frac{bx}{1+ax} > p_m \frac{by}{1+ay}$$

$\frac{x-y}{(1+ax)(1+ay)} > 0$. It implies $x > y$

Hence, $f(x)$ is non-decreasing, also, the parameters and variable involved are non-negative.

For the assumption (iii); $f'(x) = p_m \frac{b+abx-abx}{(1+ax)^2} = p_m \left[\frac{b}{(1+ax)^2}\right] > 0$, and

$f''(x) = p_m \left[\frac{-2ab(1+ax)}{(1+ax)^4}\right] = \frac{-2abp_m}{(1+ax)^3} < 0$, since all the parameters are non-negative. Therefore, (2.4) satisfies conditions (i), (ii) and (iii).

So, putting equations (2.1) and (2.4) into (2.3) gives

$$N_{t+1} = p_d p_g p_e p_m \frac{bN_t}{1+aN_t} + p_d p_g p_e p_d \left(1-p_g\right)\left(x_{t-1}+y_{t-1}\right) \quad (2.5)$$

Therefore, adopting equation (2.3) in (2.5) gives

$$N_{t+1} = p_d p_g p_e p_m \frac{bN_t}{1+aN_t} + p_d \left(1-p_g\right) N_t \quad (2.6)$$

In a more compact form (2.6) becomes

$$N_{t+1} = \frac{rbN_t}{1+aN_t} + \gamma N \quad (2.7)$$

And subsequently written as

$$N_{t+1} = \frac{\beta N_t}{1+aN_t} + \gamma N_t \quad (2.8)$$

(2.8) is a non-linear difference equation for the homogeneous population density of established mature weeds in the time-step or year $(t+1)$.

Where

$$r = P_d P_g P_e P_m, \quad \beta = rb \ , \quad \gamma = P_d(1-P_g), \quad 0 \le r \le 1, \ \beta > 0 \quad (2.9)$$

$\frac{\beta}{1+aN_t}$ represents the density-dependent net recruitment rate from generation to generation. While the term $\frac{\beta N_t}{1+aN_t}$, a saturation function gives the population growth of the weed as a function of mature weed density. As the density increases the residual seeds in the top soil are denied access to enough growth resources (e.g light, nutrient, water and micro site space area), thereby reduces the chances of seed germination and establishment. At low density, more seeds would have access to the growth resources and the proliferation of weeds follows. Therefore, Equation (2.8) gives a discrete-time model for homogenous population of single weed proliferation without control.

3.0 Stability Analysis of the Model Equation

3.1 Steady-State Solutions

Usually the first step to take in order to study the dynamics of any system (model) is to find the steady- state solutions of the system which are also the critical points (Cushing and Yicang, 1994; Bozkurt and Peker, 2014). A point is assumed to be a solution of the steady-state of the model equations only if all of its components are non-negative for biological and ecological significance. To solve for the steady state of (2.8),

Let $N_{t+1} = N_t$, implies that $\Delta N = N_{t+1} - N_t = 0$

So, $N_{t+1} = N_t = \overline{N}$ (3.1)

There are two nonnegative solutions of the steady-states for the model equation (2.8). So applying (3.1), the steady-state of (3.8) satisfies the equation

$$\overline{N} = \frac{\beta \overline{N}}{1 + a\overline{N}} + \gamma \overline{N} \tag{3.2}$$

From equation (3.2) we have

$$\overline{N}\left(1 - \frac{\beta}{1 + a\overline{N}} - \gamma\right) = 0 \tag{3.3}$$

One of the steady state solutions is $\overline{N} = 0$. The non-zero steady-state occurs when $F(\overline{N}) = N$, which is also equivalent to

$$1 - \gamma = \frac{\beta}{1 + a\overline{N}} \tag{3.4}$$

$$(1 + a\overline{N})(1 - \gamma) = \beta \tag{3.5}$$

It implies that

$$\overline{N} = \frac{\beta - (1 - \gamma)}{a(1 - \gamma)} \tag{3.6}$$

So, the second steady-state (3.6) exists and positive provided $\gamma < 1$, since $\beta > 1 - \gamma$. Hence, the two non-negative steady-states are $E_1(0)$ and $E_2\left(\frac{\beta - (1-\gamma)}{a(1-\gamma)}\right)$.

The zero state E_1 is comparable to a situation of weed density extinct (or dies out) during dry season or clearing of arable field in preparation for crop planting. While the non-zero state E_2 is liken to the existence of weeds or infestation of weed in arable field.

3.2 Local Stability of the Steady-States

To test for the local stability we adopted the well known stability theorem for discrete-time one-dimensional population models as stated in (Paul, 2007)

Theorem 1 If $f(x)$ (a model function) is differentiable then, a population model is locally asymptotically stable if $\left|f'(\overline{x})\right| < 1$ and if the model is locally stable then $\left|f'(\overline{x})\right| \leq 1$. Here $\overline{x}$ is the unique equilibrium point of function $x_{t+1} = f(x_t)$.

So, Let

$$f(\overline{N}) = \frac{\beta \overline{N}}{1 + a\overline{N}} + \gamma \overline{N} \tag{3.7}$$

Then

$$f'(\overline{N}) = \frac{(1 + a\overline{N})\beta - \beta a\overline{N}}{(1 + a\overline{N})^2} + \gamma \tag{3.8}$$

We simplified to obtained

$$f'(\bar{N}) = \frac{\beta}{(1+a\bar{N})^2} + \gamma \tag{3.9}$$

Stability of $E_1(0)$

Evaluating equation (3.9) at $N = 0$ gives

$$f'(0) = \beta + \gamma \tag{3.10}$$

So, zero steady states is locally stable if

$$\left| f'(0) \right| = \beta + \gamma < 1 \tag{3.11}$$

$$\left| f'(\bar{N}) \right| = \frac{(1-\gamma)}{\beta} < 1 \tag{3.12}$$

Otherwise, it is not stable. Thus, if the model is stable, the density of the mature weed tends to zero and the weed population dies out or eradicated. Clearly $\beta + \gamma > 1$, so $E_1 = 0$ is unstable. This implies that the density of the mature weed tends to a new equilibrium density E_2.

Stability of the solution $E_2(\bar{N})$

Evaluating equation (3.9) at $\bar{N} = \frac{\beta + \gamma - 1}{a(1-\gamma)}$ gives

$$f'(\bar{N}) = \frac{\beta}{\left(1 + a\frac{\beta + \gamma - 1}{a(1-\gamma)}\right)^2} + \gamma \tag{3.13}$$

$$f'(\bar{N}) = \frac{\beta(1-\gamma)^2}{\beta^2} + \gamma \tag{3.14}$$

$$f'(\bar{N}) = \frac{(1-\gamma)^2}{\beta} + \gamma \tag{3.15}$$

So, the steady state E_2 is stable whenever $\frac{(1-\gamma)^2}{\beta} + \gamma < 1$

That is

$$\frac{(1-\gamma)^2}{\beta} < 1 - \gamma \tag{3.16}$$

Implies

$$\frac{(1-\gamma)}{\beta} < 1 \tag{3.17}$$

Proposition 3.1

If $\beta+\gamma>1$, then the non-zero steady-state E_2 is locally stable, otherwise it is not stable.

Proof

Suppose $E_1(0)$ is stable. It implies that $\beta+\gamma<1$.

For E_2 to be stable, using T**heorem 1**, equation (3.17) must hold. That is

$$\left|\frac{1-\gamma}{\beta}\right|<1 \tag{3.18}$$

$$-1<\frac{1-\gamma}{\beta}<1 \tag{3.19}$$

$$-\beta<1-\gamma<\beta \tag{3.20}$$

$$1-\beta<\gamma<1+\beta \tag{3.21}$$

Implies

$$1<\beta+\gamma<1+2\beta \tag{3.22}$$

This completes the proof. Hence, the nonzero steady-state E_2 is locally stable. While $E_1(0)$ is unstable. Thus the density of mature weeds N_t approaches or converges (settles down) to a positive constant value given by

$$\frac{\beta+\gamma-1}{a(1-\gamma)}.$$

3.3 Global Stability of the Steady-State

It is important to know whether or not a model is globally stable. Models having this property are predictable, while those that do not can exhibit unexpected behaviour (Heinschel, 1994). One of the tools used to prove global stability in difference equations is the Schwarzian derivative, which was first introduced into the study of one-dimensional dynamical system by David Singer (Heinschel, 1994, Eduarodo, 2007). The Schwarzian derivative is given as

$$S(f,x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \tag{3.23}$$

Calculation of the Schwarzian for Model Equation (2.8)

At the steady-state (2.8) becomes

$$f(\overline{N})=\frac{\beta\overline{N}}{1+a\overline{N}}+\gamma\overline{N} \tag{3.24}$$

$$f'(\overline{N})=\frac{\beta}{(1+a\overline{N})^2}+\gamma, \tag{3.25}$$

$$f''(\bar{N}) = \frac{-2a\beta}{(1+a\bar{N})^3} \tag{3.26}$$

$$f'''(\bar{N}) = \frac{6a^2\beta}{(1+a\bar{N})^4} \tag{3.27}$$

So

$$\mathbf{S}(f,\bar{N}) = \frac{6\beta a^2}{(1+a\bar{N})^2[\beta+\gamma(1+a\bar{N})^2]} - \frac{3}{2}\left[\frac{-2a\beta}{(1+a\bar{N})[\beta+\gamma(1+a\bar{N})^2]}\right]^2$$

$$= \frac{6a^2\beta[\beta+\gamma(1+a\bar{N})^2] - 6a^2\beta^2}{\left\{(1+a\bar{N})[\beta+\gamma(1+a\bar{N})^2]\right\}^2}$$

$$\mathbf{S}(f,\bar{N}) = \frac{6a^2\beta\gamma(1+a\bar{N})^2}{\left\{(1+a\bar{N})[\beta+\gamma(1+a\bar{N})^2]\right\}^2} \tag{3.28}$$

this gives

$$\mathbf{S}(f,\bar{N}) = \frac{6a^2\beta\gamma}{[\beta+\gamma(1+a\bar{N})^2]^2} \tag{3.29}$$

This show that, $\mathbf{S}(f,\bar{N}) > 0$ everywhere. Hence, non-zero steady-state (E_2) is not globally stable. Knowing whether or not the model is globally stable gives additional understanding of the behaviour of a model. Hence, the weed population may exhibit unexpected behaviours (that is the population may not be predictable).

4.0 Graphical Profile of the formulated Model equation

The model equation (2.8) is used to simulate the population density dynamics of single weed using the parameter values $\beta = 1.4$, , $a = 0.05$ $\gamma = 0.8$ from one time step (generation) to the next. Shown in figures (4.1) and (4.2)

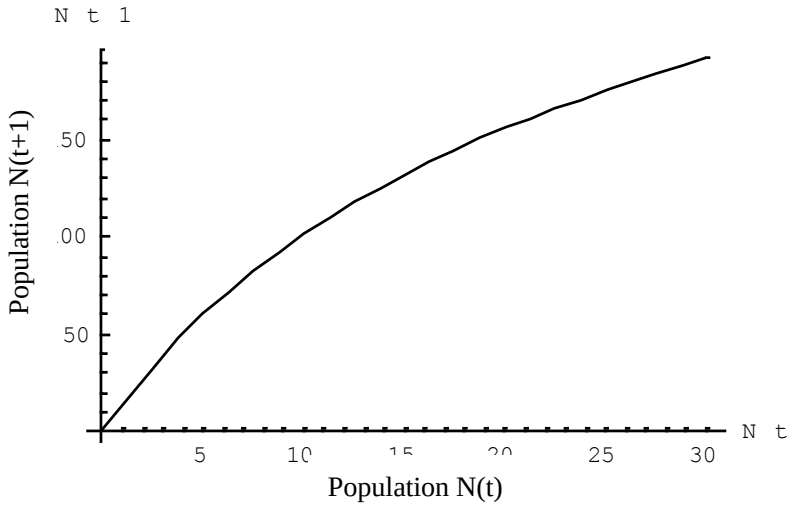

Figure 4.1 Dynamics of Population density of annual weed from t to $(t+1)$.
It represents the population density dynamics of weeds from one generation

(time step) to another, if there was no control and all necessary growth resources are available.

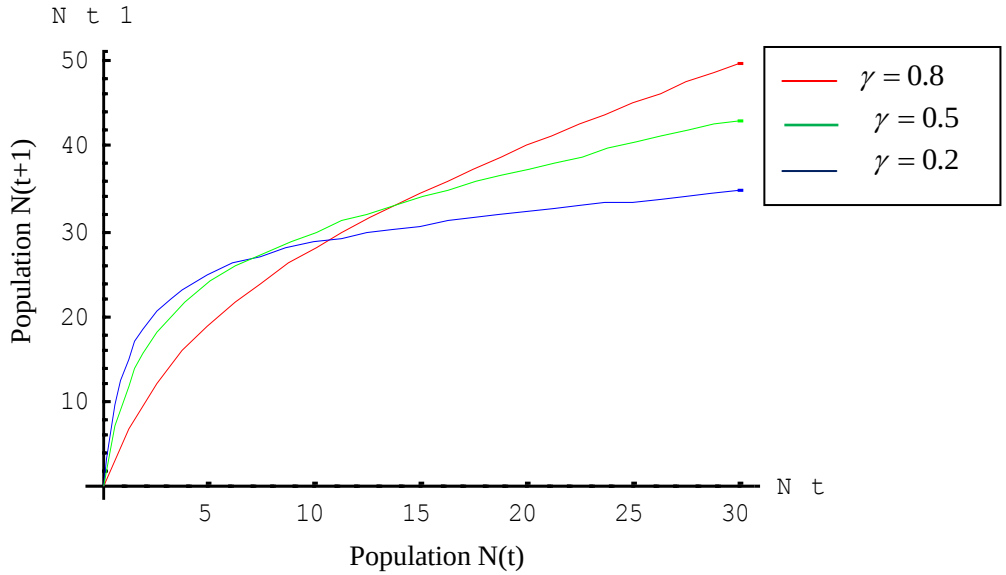

Figure 4.2 Effect of seed dormancy rate on the density of from t to $(t+1)$.

The figure 4.2 depicts the effect of changing the dormancy rate (γ) of seed bank seeds of weed on its population density in the next time step. The lower the dormancy rate the faster the weed density approaches its maximum population which the available space could accommodate.

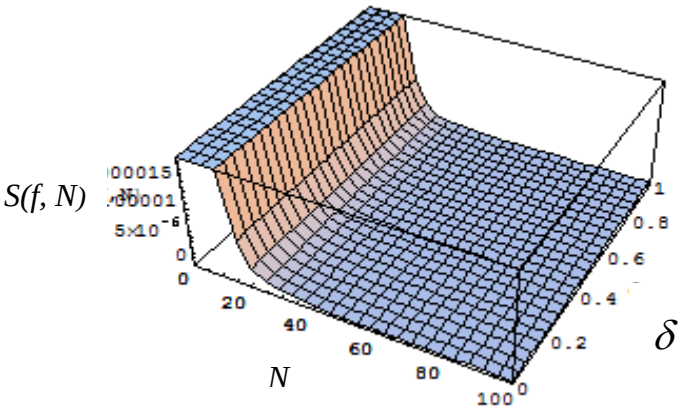

Figure 4.3 Schwarzian of model (2.8) with no control

Figure 4.3 shows that Schwarzian is positive everywhere since its graph has no evident stumpy region. So, we conclude that the weed proliferation may exhibits an unpredictable behaviour.

5.0 Conclusion and Further study

The steady-state solutions of the proposed model equation were obtained and analysed for local and global stabilities. The analysis shows that our model is locally asymptotically stable but globally unstable. This result is contrary to the interesting property of the most standard biological one-dimensional discrete models, which display global stability if they are locally stable. Although, our model equation falls within the category of population models that exhibit local stability but not globally stable. Therefore, the model may exhibit unexpected behaviours and the population density of the weed may not be predictable. Further study of the behaviour of this model for global stability is under consideration.

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